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# EXISTENCE OF MÖBIUS ENERGY MINIMIZERS IN CONSTRAINED SETTINGS

## 1 Introduction

In the late 1980s and early 1990s, Fukuhara [3], Sakuma [8], and O’Hara [6] developed various types of knot energies based on the electrostatic potential of uniformly charged curves in  $\mathbb{R}^3$ . Within this setting, it is natural to seek an optimal embedding of a knot in  $\mathbb{R}^3$  as a minimal-energy representative of the knot class. Further, the energy of a knotted curve should approach  $+\infty$  as the curve obtains a self-intersection, so as to prevent the curve from changing to a different knot type. The so-called “Möbius energy” is one of O’Hara’s knot energies which has the above property of self-repulsiveness. Freedman, He, and Wang [2] showed that, in a given knot type in  $\mathbb{R}^3$ , there exists a representative with minimal Möbius energy, which is embedded and  $C^{1,1}$ . Z. He [5] later verified that this representative is in fact  $C^\infty$ .

Self-intersecting curves play a fundamental role in regular homotopy theory [9], but in the context of optimization, they have infinite Möbius energy. In the spirit of a Ginzburg-Landau problem [1], the current goal is to understand geometric properties of certain planar curves which self-intersect, but nevertheless represent local minima for a corresponding renormalized energy. The main purpose of this note is to discuss the existence of “Möbius energy minimizers” to particular classes of bounded planar curves which are constrained to have some prescribed self-intersection.

## 2 Background

This section is devoted to the basic framework of the Möbius energy minimization for embedded knots (simple loops) in  $\mathbb{R}^3$ , as laid out in [2].

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## 2.1 Definition and Consequences

**Definition 2.1.** Let  $\gamma$  be a curve in  $\mathbb{R}^3$  given by  $\gamma(x)$ , where  $x$ -values are taken from some subinterval  $X$  of  $\mathbb{R}$  or of  $\mathbb{S}^1$ . Also assume that  $|\dot{\gamma}|$  is locally integrable. Denote by  $D(\gamma(x), \gamma(y))$  the length of the shorter arc along the curve  $\gamma$  between  $\gamma(x)$  and  $\gamma(y)$ . Define the (Möbius) energy  $E(\gamma)$  to be

$$E(\gamma) = \iint_{X \times X} \left( \frac{1}{|\gamma(x) - \gamma(y)|^2} - \frac{1}{D(\gamma(x), \gamma(y))^2} \right) |\dot{\gamma}(x)| |\dot{\gamma}(y)| dx dy.$$

**Remark 2.2.** *It is clear from this definition that  $E(\gamma)$  is always non-negative and that  $E(\gamma)$  is invariant under rigid motions, linear rescaling, and reparametrization.*

**Remark 2.3.** *Intuitively,  $E$  is self-repulsive. That is, if  $\{\gamma_i\}_{i=1}^{\infty}$  is a sequence of curves which converges pointwise to a self-intersecting curve, then  $E(\gamma_i) \rightarrow +\infty$ .*

This second remark is formalized by the following result.

**Theorem 2.4** ([7] Theorem 2.3, [2] Lemma 1.2). *Let  $\gamma$  be a constant-speed curve with finite length and finite energy. Then the function  $\gamma : X \rightarrow \mathbb{R}^3$  is a bi-Lipschitz embedding. Furthermore, the bi-Lipschitz constant is bounded above by a constant depending only on  $E(\gamma)$ .*

## 2.2 Minimization for embedded knots

The key result for simple loops is the following.

**Theorem 2.5** ([2], Theorem 4.3). *Let  $K$  be an irreducible knot, that is, a knot which cannot be represented as the connected sum of two non-trivial knots. There exists a simple loop  $\gamma_K : \mathbb{S}^1 \rightarrow \mathbb{R}^3$  of the knot type  $K$  such that  $E(\gamma_K) \leq E(\gamma)$  for any simple loop  $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^3$  of that knot type.*

PROOF. The proof of this result has two broad components. The first is a compactness property ([2], Lemma 4.2) which relies on the earlier theorem concerning bi-Lipschitz embeddings of finite-energy curves. Given a sequence  $\{\gamma_i\}_{i=1}^{\infty}$  of simple loops of knot type  $K$  approaching the infimal energy of the knot type, compactness guarantees the existence of a subsequence converging uniformly to a simple loop  $\gamma_{\infty}$  such that

$$E(\gamma_{\infty}) \leq \inf\{E(\gamma) \mid \gamma \text{ is a loop of knot type } K\}.$$

However, the minimizing sequence  $\{\gamma_i\}_{i=1}^{\infty}$  may degenerate so that the limit loop  $\gamma_{\infty}$  is not of knot type  $K$ .

The second component of the proof corrects this problem. Möbius energy is so named because the energy of a knot is invariant under Möbius transformations of the one-point compactification of  $\mathbb{R}^3$  ([2], Theorem 2.1). Applying particular Möbius transformations to the minimizing sequence prevents any such degeneration of the knot type, thus allowing a (potentially new) limit loop  $\tilde{\gamma}_\infty$  to be chosen within the knot type  $K$ . Setting  $\gamma_K := \tilde{\gamma}_\infty$  finishes the proof.  $\square$

### 3 Self-Intersection

The problem at hand is how to find an optimal immersion of a self-intersecting closed curve in the plane; given an isotopy class  $I$  of closed curves in  $\mathbb{R}^2$  with prescribed self-intersection, is there a curve  $\gamma_I \in I$  such that  $E(\gamma_I) \leq E(\gamma)$  for all  $\gamma \in I$ ? The question as stated is empty, as every curve  $\gamma$  in the class  $I$  has infinite energy. A renormalization of the Möbius energy  $E$  is necessary for a proper formulation of the problem. The approach is to first find a minimizer of a renormalized energy that disregards the crossing in a manner dependent on some parameter, then to realize an optimal immersion as a limit of these minimizers as the parameter degenerates. Again, this technique is comparable to minimization in a Ginzburg-Landau setting [1].

To begin, let  $\gamma : X \rightarrow \mathbb{R}^2$  be a self-intersecting closed curve. For the purposes of this note, assume that  $\gamma$  has finite length  $L = L(\gamma)$  and intersects itself only once, at the origin. The curve  $\gamma$  could be a figure eight shape, for example. Denote by  $u, v \in X$  the distinct times that yield  $\gamma(u) = \gamma(v) = (0, 0) \in \mathbb{R}^2$ . Let  $\lambda = \lambda(\gamma) = \frac{D(\gamma(u), \gamma(v))}{L(\gamma)}$ , the portion of the length of  $\gamma$  taken up by the shorter of its two distinct “loops.” Note that  $0 < \lambda \leq 1/2$ .

**Definition 3.1.** With  $\gamma$ ,  $u$ ,  $v$ ,  $L$ , and  $\lambda$  as above, choose  $\delta_0 \ll L\lambda$ . Let

$$X_{\delta_0} = \{x \in X \mid D(\gamma(x), \gamma(u)), D(\gamma(x), \gamma(v)) \geq \delta_0\}.$$

Define  $E_{\delta_0}(\gamma)$  to be the energy of  $\gamma$  taken over pairs  $(x, y)$  with both  $x$  and  $y$  “ $\delta_0$  away from the crossing,” that is

$$E_{\delta_0}(\gamma) = \iint_{X_{\delta_0} \times X_{\delta_0}} \left( \frac{1}{|\gamma(x) - \gamma(y)|^2} - \frac{1}{D(\gamma(x), \gamma(y))^2} \right) |\dot{\gamma}(x)| |\dot{\gamma}(y)| dx dy.$$

**Remark 3.2.** Following Remark 2.2,  $E_{\delta_0}$  is non-negative and invariant under rigid motions and reparametrization. It also follows that  $E_{\delta_0}(\gamma) = E_{s\delta_0}(s\gamma)$  for any scalar  $s$ .

The main result of this note is the existence of a curve which minimizes the renormalized energy  $E_{\delta_0}$  among a particular class of curves.

**Theorem 3.3.** *Choose  $\lambda \in (0, \frac{1}{2}]$ . Let  $I$  be an isotopy class of closed curves in  $\mathbb{R}^2$  with one self-intersection, and let*

$$\mathcal{F} = \{ \gamma \in I \mid L(\gamma) = 1, \lambda(\gamma) = \lambda, \gamma \text{ crosses itself at the origin} \}.$$

*For every  $\delta_0 \ll \lambda$ , there exists a curve  $\gamma_{\delta_0} \in \mathcal{F}$  such that  $E_{\delta_0}(\gamma_{\delta_0}) \leq E_{\delta_0}(\gamma)$  for all  $\gamma \in \mathcal{F}$ .*

PROOF. The details appear in [4]. The key step is a variation of Theorem 2.4 which shows that for a unit-speed curve  $\gamma \in \mathcal{F}$  with  $E_{\delta_0}(\gamma) < \infty$ , the restricted map  $\gamma : X_{\delta_0} \rightarrow \mathbb{R}^2$  is a bi-Lipschitz embedding. In this case, the bi-Lipschitz constant is bounded above by  $\frac{24}{\lambda^2} \exp[(49/40)E_{\delta_0}(\gamma)]$ . A standard compactness argument completes the proof.  $\square$

The current work of [4] is to determine what happens to the minimizers  $\gamma_{\delta_0}$  as  $\delta_0 \rightarrow 0$ . In order to show the convergence (away from the crossing) for a sequence  $\{\gamma_{\delta_i}\}_{i=1}^{\infty}$  with  $\delta_i \rightarrow 0$ , introduce a new parameter  $\eta > \delta_i$  for all  $i$  with  $\eta \ll \lambda$ . For each such  $\eta$ , the value of  $E_{\eta}(\gamma_{\delta_i})$  and other related quantities can be estimated *independently of  $i$* .

In addition to the above limit procedure, [4] will also include the natural generalizations of the optimization problem described in this paper: allowing  $L$  and  $\lambda$  to vary within the problem, relaxing the curves from the plane into  $\mathbb{R}^3$ , and increasing the number of crossings from 1 up to  $n$ . For the general  $n$ -crossing problem, there are two cases to consider; the location of the crossings in  $\mathbb{R}^2$  may be prescribed, or they may be free to move.

An alternate approach to §3 is to realize an optimal planar immersion as the limit of optimally embedded curves in  $\mathbb{R}^3$ . More specifically, use the methods of [2] to find a loop that minimizes the Möbius energy among all loops of the same knot type, under the constraint that the loops are all contained in the slab  $\mathbb{R}^2 \times [0, \varepsilon]$ . Then take the limit of these “ $\varepsilon$ -minimizers” as  $\varepsilon \rightarrow 0$ . However, it remains to be shown that a loop  $\gamma$  in  $\mathbb{R}^3$  can undergo a Möbius transformation  $T$  to place it in the  $\varepsilon$ -slab with the length of  $T \circ \gamma$  sufficiently bounded away from 0 to prevent degeneration of the continuum of the “ $\varepsilon$ -minimizers.”

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## References

- [1] Fabrice Bethuel, Haïm Brezis, Frédéric Hélein, *Ginzburg-Landau Vortices*, Progr. Nonlinear Differential Equations Appl., **13** Birkhuser Boston, Inc., Boston, MA, 1994.
- [2] Michael Freedman, Zheng-Xu He, Zhenghan Wang, *Möbius energy of knots and unknots*, Ann. of Math. (2), **139** (1994), 1–50.
- [3] S. Fukuhara, *Energy of a knot*, A Fête of Topology, 443–451, (Y. Matsumoto, T. Mizutani, S. Morita, eds.), Academic Press, Boston, 1988.
- [4] Ryan Dunning, *Asymptotics under self-intersection for minimizers of self-avoiding energies*, Ph.D. Thesis, Rice University, in preparation.
- [5] Zheng-Xu He, *The Euler-Lagrange equation and heat flow for the Möbius energy*, Comm. Pure Appl. Math., **53** (2000), 399–431.
- [6] Jun O’Hara, *Energy of a knot*, Topology, **30** (1991), 241–247.
- [7] Jun O’Hara, *Family of energy functionals of knots*, Topology Appl., **48** (1992), 147–161.
- [8] M. Sakuma, *Problem no. 8 (in Japanese)*, The collection of problems on “Low dimensional topology and related matters” (S. Kojima, S. Negami, eds.), (1987), 7.
- [9] Hassler Whitney, *On regular closed curves in the plane*, Compos. Math., **4** (1937), 276–284.